Metode Pemisahan Variabel

- $u(x,y) = F(x).G(y)$
- $u(x,t) = F(x).G(t)$

- F dan G dicari dari persamaan diferensial yang ada beserta syarat-syarat yang harus dipenuhi oleh fungsi-fungsi tersebut

Contoh : Carilah penyelesaian persamaan diferensial berikut dengan cara pemisahan variabel $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Penyelesaian :

$$u(x, y) = F(x).G(y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[F(x).G(y)\right] = \frac{\partial F}{\partial x} G = F'G$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[F(x).G(y)\right] = F \frac{\partial G}{\partial x} = G'F$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} \quad \rightarrow \quad F'G = -FG'$$

$$\frac{F'}{F} = -\frac{G'}{G} = k \qquad k = \text{ konstanta}$$

$$\frac{F'}{F} = k \quad \Rightarrow \quad F' = Fk \quad \Rightarrow F' - kF = 0$$

$$\lambda - k = 0 \quad \Rightarrow \quad \lambda = k \quad \Rightarrow F(x) = Ae^{kx}$$

$$-\frac{G'}{G} = k \quad \Rightarrow \quad -G' = Gk \quad \Rightarrow -G' - kG = 0$$

$$-\lambda - k = 0 \quad \Rightarrow \quad \lambda = -k \quad \Rightarrow G(y) = Be^{-ky}$$

$$\text{Jadi } u(x, y) = F(x).G(y) = Ae^{kx}.Be^{-kx} = ABe^{k(x-y)} = Ce^{k(x-y)}$$

Contoh : Carilah penyelesaian persamaan diferensial berikut dengan cara pemisahan variabel $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \quad u(0,y) = 8e^{-3y}$

Penyelesaian :

$$u(x,y) = F(x).G(y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[F(x).G(y)\right] = \frac{\partial F}{\partial x}G = F'G$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[F(x).G(y)\right] = F\frac{\partial G}{\partial x} = FG'$$

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \rightarrow \quad F'G = 4FG'$$

$$\frac{F'}{4F} = \frac{G'}{G} = k \qquad k = \text{ konstanta}$$

$$\frac{F'}{4F} = k \quad \Rightarrow \quad F' = 4kF \quad \Rightarrow F' - 4kF = 0$$

$$\lambda - 4k = 0 \quad \Rightarrow \quad \lambda = 4k \quad \Rightarrow F(x) = Ae^{4kx}$$

$$\frac{G'}{G} = k \quad \Rightarrow \quad G' = Gk \quad \Rightarrow G' - kG = 0$$

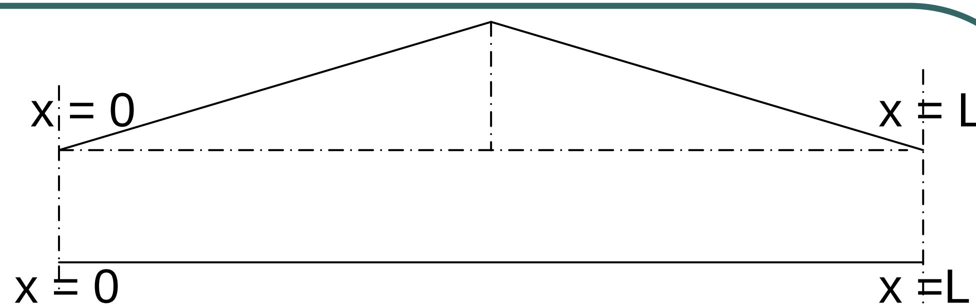
$$\lambda - k = 0 \quad \Rightarrow \quad \lambda = k \quad \Rightarrow G(y) = Be^{ky}$$

$$\text{Jadi } u(x,y) = F(x).G(y) = Ae^{4kx}.Be^{kx} = ABe^{k(4x+y)} = Ce^{k(4x+y)}$$

$$\text{Syarat batas :} \quad u(0,y) = Ce^{ky} = 8e^{-3y} \quad \Rightarrow \; C = 8, \quad y = -3$$

$$\text{Jadi } u(x,y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

Contoh : $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



Penyelesaian : $u(x,t) = F(x).G(t)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left[F(x).G(y)\right]\right\} = F\,\frac{\partial}{\partial t}\left[\frac{\partial G}{\partial t}\right] = FG''$$

Langkah I

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left[F(x).G(y)\right]\right\} = \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial x}\right]G = F''G$$

$FG'' = c^2 F''G$ dibagi dengan $c^2 FG$

$$\frac{G''}{c^2 G} = \frac{F''}{F} = k \qquad k = \text{ konstanta}$$

$$\Rightarrow F'' - kF = 0$$

$$\Rightarrow G'' - c^2 kG = 0$$

Syarat Batas

$$u(0,t) = F(0).G(t) = 0, \qquad u(L,t) = F(L).G(t) = 0$$

untuk seluruh t

Langkah II

Penyelesaian $F'' - kF = 0$

$$F(0) = 0, \qquad F(L) = 0$$

Pilih $k = -p^2$ (*p = 0, p > 0 dan p < 0*)

$$F'' + p^2 F = 0, \qquad \Rightarrow \qquad F(x) = A\cos px + B\sin px$$

$$F(0) = A = 0, \qquad F(L) = B\sin pL = 0$$

$$F_n(x) = \sin\frac{n\pi}{L}x, \qquad n = 1,2,\cdots$$

Penyelesaian $G'' - c^2 kG = 0$

Pilih $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$$G'' - \lambda_n{}^2 G = 0, \qquad \lambda_n = \frac{cn\pi}{L}$$

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$$

$$u(x,t) = F_n(x).G_n(t)$$

$$= \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x, \qquad n = 1,2,\cdots$$

Langkah III Penyelesaian dengan Deret Fourier

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\sin\frac{n\pi}{L}x$$

Syarat awal $u(x,0)$ *initial displacement*

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x)$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \qquad n=1,2,\cdots$$

Syarat awal $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ *initial velocity*

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}\left(-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n \sin\lambda_n t\right)\sin\frac{n\pi}{L}x\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi}{L}x = g(x)$$

$$B_n^*\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \qquad \lambda_n = \frac{cn\pi}{L}$$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \qquad n = 1,2,\cdots$$

$\text{Bila kecepatan awal } (\textit{initial velocity}) \; g(x) = 0 \; \text{ maka } \; B_n^* = 0$

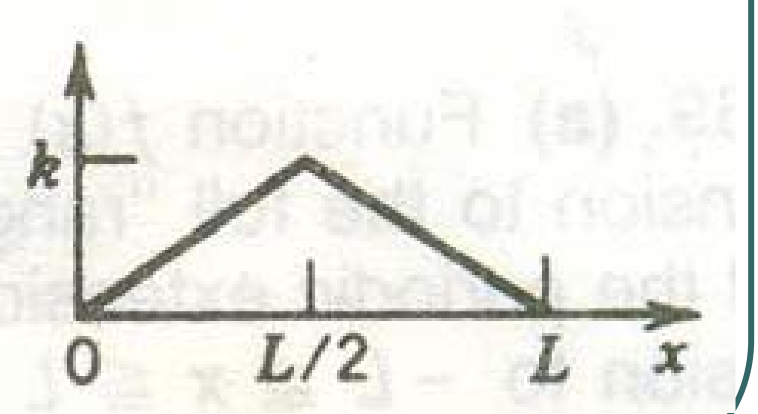
dengan demikian penyelesaian persaman gelombang menjadi :

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x \qquad \lambda_n = \frac{cn\pi}{L}$$

Contoh :

- Carilah penyelesaian persamaan gelombang dengan simpangan awal (*initial deplasement*)

$$f(x) = \begin{cases} \dfrac{2k}{L} x & \text{jika} \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{jika} \quad \dfrac{L}{2} < x < L \end{cases}$$



dan kecepatan awal adalah nol.

Penyelesaian :

Karena kecepatan awal (*initial velocity*) $g(x)=0$ maka $B_n^* = 0$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \qquad n = 1, 2, \cdots$$

$$B_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right]$$

$$B_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\cos\lambda_n t\sin\frac{n\pi}{2}x$$

$$u(x,t) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t + \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + \cdots\right)$$