



# Perkalian dan Pembagian Bilangan Kompleks dalam bentuk Kutub

Perkalian

$$\begin{aligned}z_1 z_2 &= r_1 (\cos \theta_1 + j \sin \theta_1) r_2 (\cos \theta_2 + j \sin \theta_2) \\&= r_1 r_2 (\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)) \\&= r_1 r_2 \angle \theta_1 + \theta_2\end{aligned}$$

Pembagian

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1 (\cos \theta_1 + j \sin \theta_1)}{r_2 (\cos \theta_2 + j \sin \theta_2)} \\&= \frac{r_1 (\cos \theta_1 + j \sin \theta_1) (\cos \theta_2 - j \sin \theta_2)}{r_2 (\cos \theta_2 + j \sin \theta_2) (\cos \theta_2 - j \sin \theta_2)}\end{aligned}$$



$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + j \sin \theta_1)(\cos \theta_2 - j \sin \theta_2)}{r_2(\cos \theta_2 + j \sin \theta_2)(\cos \theta_2 - j \sin \theta_2)}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 \cos \theta_2 + j \sin \theta_1 \cos \theta_2 - j \cos \theta_1 \sin \theta_2 + \sin \theta_1 \sin \theta_2)}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)} \\ &= \frac{r_1(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + j(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{r_2(1)} \\ &= \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + j(\sin(\theta_1 - \theta_2))) \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

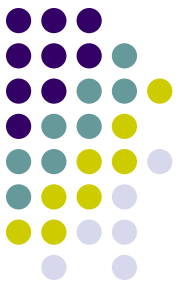
Contoh :

$$z_1 = 10(\cos 60^\circ + j \sin 60^\circ) = 10 \angle 60^\circ$$

$$z_2 = 5(\cos 40^\circ + j \sin 40^\circ) = 5 \angle 40^\circ$$

$$\begin{aligned} z_1 z_2 &= 10(\cos 60^\circ + j \sin 60^\circ) 5(\cos 40^\circ + j \sin 40^\circ) \\ &= 50(\cos 100^\circ + j \sin 100^\circ) = 50 \angle 100^\circ \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{10(\cos 60^\circ + j \sin 60^\circ)}{5(\cos 40^\circ + j \sin 40^\circ)} \\ &= 2(\cos 20^\circ + j \sin 20^\circ) = 2 \angle 20^\circ \end{aligned}$$



# Perpangkatan Bilangan Kompleks Bentuk Kutub



$$z_1 = r(\cos \theta + j \sin \theta) = r \angle \theta$$

$$z_2 = r(\cos \theta + j \sin \theta) = r \angle \theta$$

$$z_2 = z_1$$

$$\begin{aligned} z_1 z_1 &= r(\cos \theta + j \sin \theta) r(\cos \theta + j \sin \theta) \\ &= r \cdot r (\cos(\theta + \theta) + j \sin(\theta + \theta)) \end{aligned}$$

$$z_1^2 = r^2 \angle 2\theta$$

Teorema DeMoivre (*ABRAHAM DE MOIVRE(1667-1754)*)

$$z^n = r^n (\cos n\theta + j \sin n\theta) = r^n \angle n\theta$$

# Penjabaran $\cos n\theta$ dan $\sin n\theta$



$$\cos 3\theta + j \sin 3\theta = (\cos \theta + j \sin \theta)^3$$

$$\cos 3\theta + j \sin 3\theta = \cos^3 \theta + 3\cos^2 \theta(j \sin \theta) + 3\cos \theta(j \sin \theta)^2 + (j \sin \theta)^3$$

$$= \cos^3 \theta + j3\cos^2 \theta \sin \theta - 3\cos \theta \sin^2 \theta - j \sin^3 \theta$$

$$= (\cos^3 \theta - 3\cos \theta \sin^2 \theta) + j(3\cos^2 \theta \sin \theta - \sin^3 \theta)$$

riil

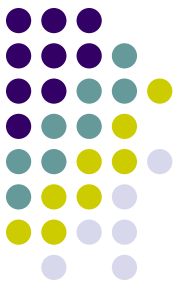
imajiner

$$\begin{aligned} \cos 3\theta &= \cos^3 \theta - 3\cos \theta \sin^2 \theta \\ \sin 3\theta &= 3\cos^2 \theta \sin \theta - \sin^3 \theta \end{aligned}$$

Samakan bagian riil dan bagian imajiner

*Jika*  $\sin^2 \theta = (1 - \cos^2 \theta)$  *maka*  $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$

*Jika*  $\cos^2 \theta = (1 - \sin^2 \theta)$  *maka*  $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$



# Penjabaran $\cos^n \theta$ dan $\sin^n \theta$ *dinyatakan dalam sinus dan cosinus kalipatan $\theta$*

$$z^n = \cos n\theta + j \sin n\theta$$

$$\frac{1}{z^n} = z^{-n} = \cos n\theta - j \sin n\theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = j2 \sin n\theta$$



Contoh : jabarkan  $\cos^3 \theta$

$$z + \frac{1}{z} = 2 \cos \theta$$

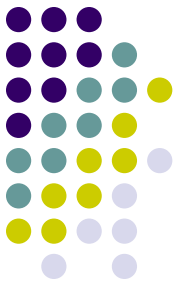
$$\begin{aligned} (2 \cos \theta)^3 &= \left( z + \frac{1}{z} \right)^3 = z^3 + 3z^2 \left( \frac{1}{z} \right) + 3z \left( \frac{1}{z^2} \right) + \frac{1}{z^3} \\ &= \left( z^3 + \frac{1}{z^3} \right) + 3 \left( z + \frac{1}{z} \right) \end{aligned}$$

$$(2 \cos \theta)^3 = 2 \cos 3\theta + 3(2 \cos \theta)$$

$$8 \cos^3 \theta = 2 \cos 3\theta + 6 \cos \theta$$

$$4 \cos^3 \theta = \cos 3\theta + 3 \cos \theta$$

$$\cos^3 \theta = \frac{1}{4} (\cos 3\theta + 3 \cos \theta)$$



# Tempat Kedudukan



Contoh :

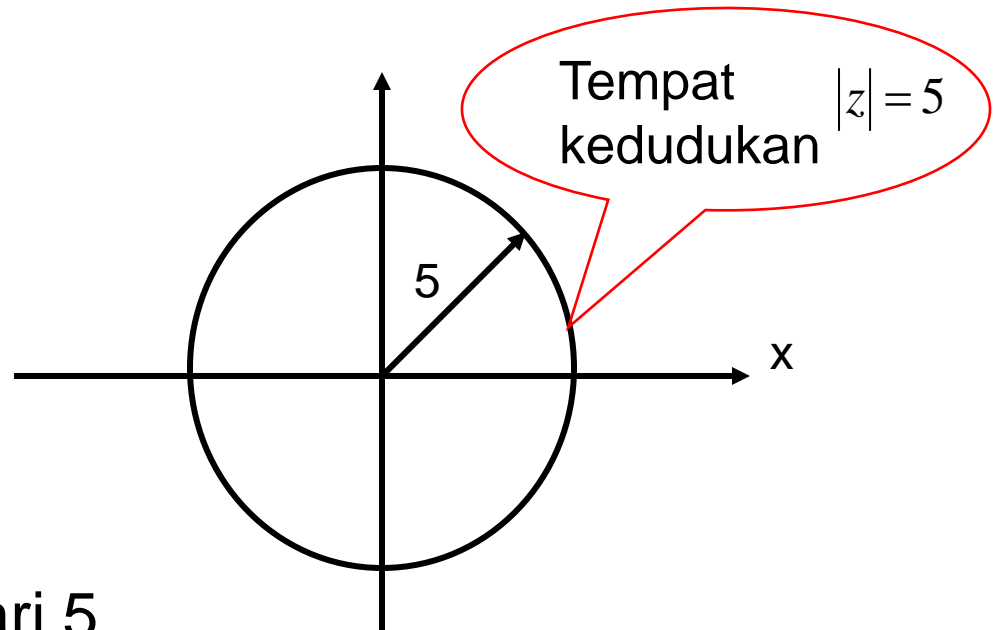
Jika  $z = x + jy$ , tentukan tempat kedudukan yang didefinisi kan oleh  $|z| = 5$

$$|z| = \sqrt{x^2 + y^2} = 5$$

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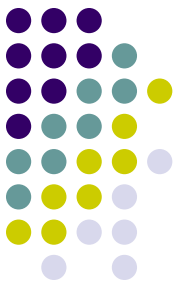
$$x^2 + y^2 = 25$$

Persamaan lingkaran yang berpusat (0,0) dan berjari-jari 5





Contoh :



Jika  $z = x + jy$ , tentukan tempat kedudukan

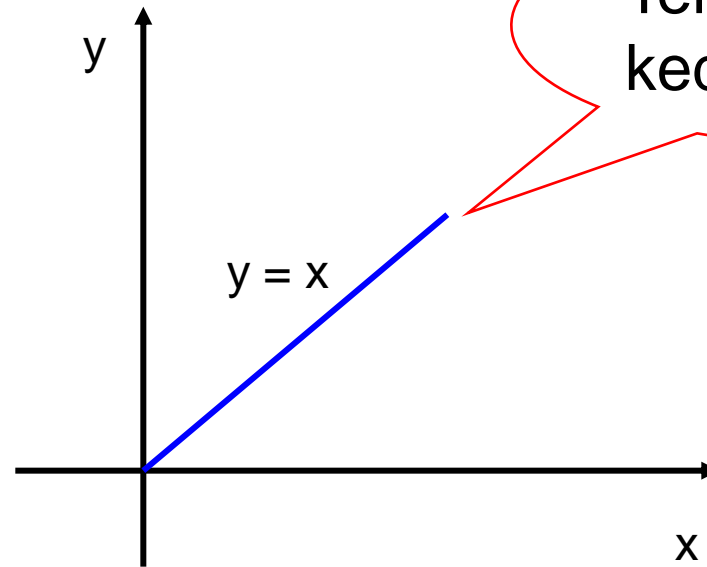
yang didefinisikan oleh  $\arg z = \frac{\pi}{4}$

$$\arg z = \tan^{-1} \frac{y}{x}$$

$$\tan^{-1} \frac{y}{x} = \frac{\pi}{4}$$

$$\frac{y}{x} = \tan \frac{\pi}{4} = \tan 45^\circ = 1$$

$$\frac{y}{x} = 1, \quad \Rightarrow y = x$$

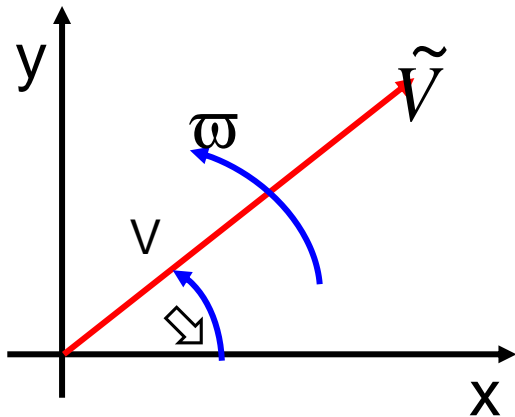


Jadi tempat kedudukannya adalah garis lurus  $y = x$  dengan  $y > 0$



# Phasor

$$v(t) = V_m \cos(\omega t + \phi) = V_m \cos(2\pi f t + \phi)$$



$V_m = \text{peak value}$

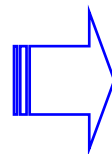
$\omega = \text{angular frequency}$

$\phi = \text{phase angle}$

Phasor tegangan dan arus

$$\tilde{V} = V \angle \phi$$

$$\tilde{I} = I \angle \phi$$



$$\tilde{V} = \tilde{I} Z$$

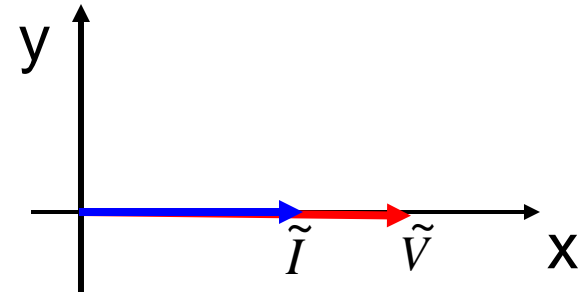
# Phasor Resistor, Induktor dan Kapasitor



Tahanan (Resistor)

$$\tilde{I} = I\angle 0, \quad Z = R\angle 0$$

$$\tilde{V} = IR\angle 0$$



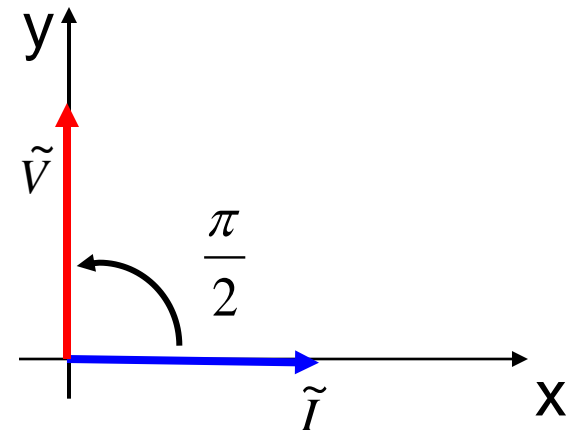
Induktor

$$\tilde{I} = I\angle 0, \quad Z = \omega L\angle \pi/2$$

$$\tilde{V} = I\omega L\angle \pi/2$$

$$Z = \omega L e^{j\pi/2} = \omega L \left( \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} \right)$$

$$= j\omega L$$





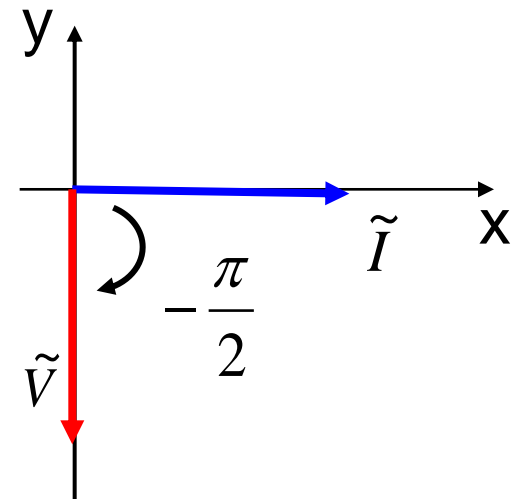
# Kapasitor

$$\tilde{I} = I \angle 0, \quad Z = \frac{1}{\omega C} \angle -\pi/2$$

$$\tilde{V} = \frac{I}{\omega C} \angle -\pi/2$$

$$Z = \frac{e^{-j\pi/2}}{\omega C} = \frac{1}{\omega C} \left( \cos \frac{\pi}{2} - j \sin \frac{\pi}{2} \right)$$

$$= -\frac{j}{\omega C} \quad \text{atau} \quad Z = \frac{1}{j\omega C}$$



# Rangkaian RLC



Hukum Kirchhoff

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_C + \tilde{V}_L$$

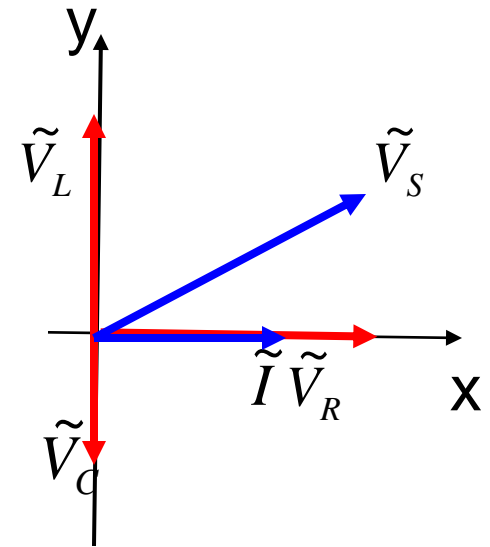
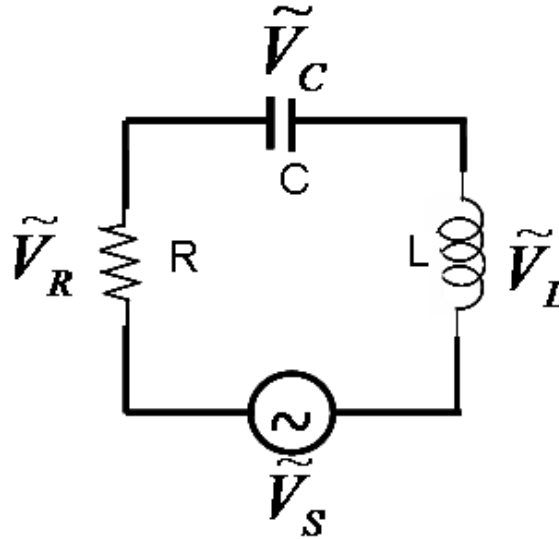
$$\tilde{V}_R = \tilde{I}R \angle 0 = \tilde{I}R$$

$$\tilde{V}_L = \tilde{I} \omega L \angle \pi/2 = j\tilde{I} \omega L$$

$$\tilde{V}_C = \frac{\tilde{I}}{\omega C} \angle -\pi/2 = \frac{\tilde{I}}{j\omega C}$$

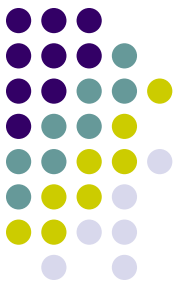
$$\tilde{V}_S = \tilde{I}R + \tilde{I}j\omega L + \frac{\tilde{I}}{j\omega C}$$

$$\tilde{V}_S = \tilde{I} \left( R + j\omega L + \frac{1}{j\omega C} \right)$$



$$Z = R + j\omega L + \frac{1}{j\omega C}$$

# Rangkuman



**TABLE 9.2** Summary of voltage-current relationships.

Element	Time domain	Frequency domain
$R$	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
$L$	$v = L \frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
$C$	$i = C \frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

**TABLE 9.3** Impedances and admittances of passive elements.

Element	Impedance	Admittance
$R$	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
$L$	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
$C$	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$



Find  $v(t)$  and  $i(t)$  in the circuit shown in Fig. 9.16.

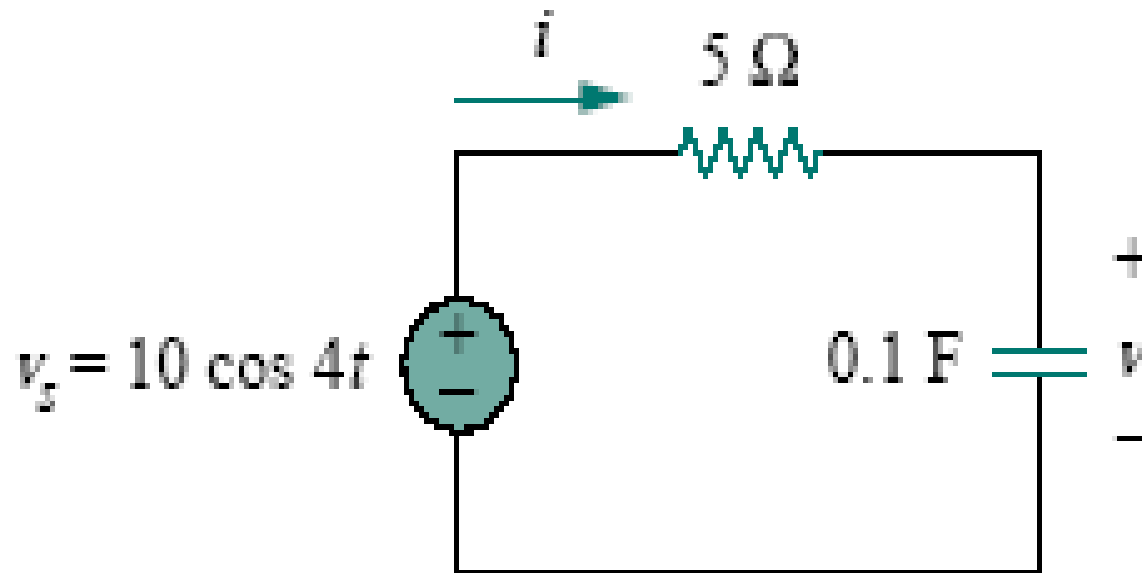


Figure 9.16

For Example 9.9.



**Solution:**

From the voltage source  $10 \cos 4t$ ,  $\omega = 4$ ,

$$\mathbf{V}_s = 10 \angle 0^\circ \text{ V}$$

The impedance is

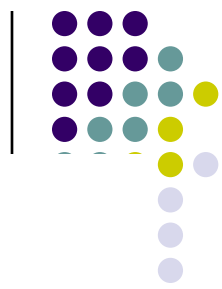
$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5 \Omega$$

Hence the current

$$\begin{aligned} \mathbf{I} &= \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10 \angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \\ &= 1.6 + j0.8 = 1.789 \angle 26.57^\circ \text{ A} \end{aligned} \quad (9.9.1)$$

The voltage across the capacitor is





$$\begin{aligned} \mathbf{V} = \mathbf{IZ}_C &= \frac{\mathbf{I}}{j\omega C} = \frac{1.789 \angle 26.57^\circ}{j4 \times 0.1} \\ &= \frac{1.789 \angle 26.57^\circ}{0.4 \angle 90^\circ} = 4.47 \angle -63.43^\circ \text{ V} \end{aligned} \quad (9.9.2)$$

Converting  $\mathbf{I}$  and  $\mathbf{V}$  in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that  $i(t)$  leads  $v(t)$  by  $90^\circ$  as expected.

Problem 8. A circuit comprises a resistance of  $90\ \Omega$  in series with an inductor of inductive reactance  $150\ \Omega$ . If the supply current is  $1.35\angle 0^\circ\ \text{A}$ , determine (a) the supply voltage, (b) the voltage across the  $90\ \Omega$  resistance, (c) the voltage across the inductance, and (d) the circuit phase angle. Draw the phasor diagram.

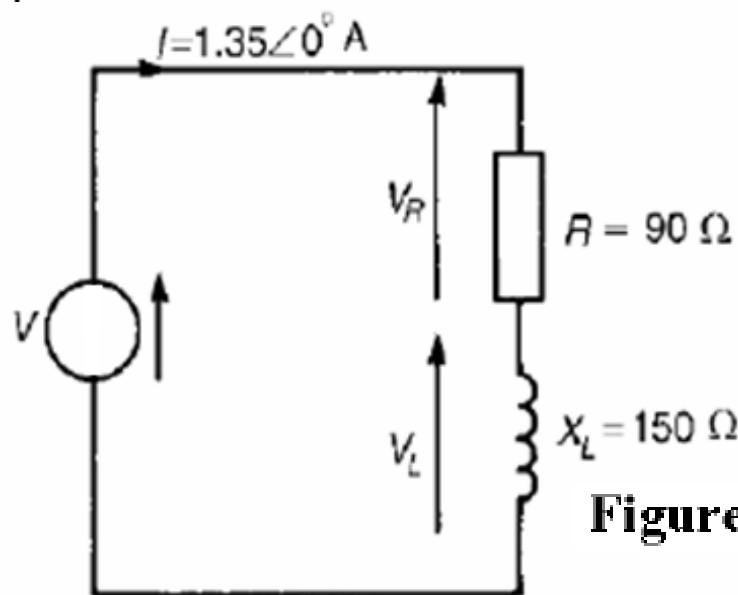


Figure 24.12

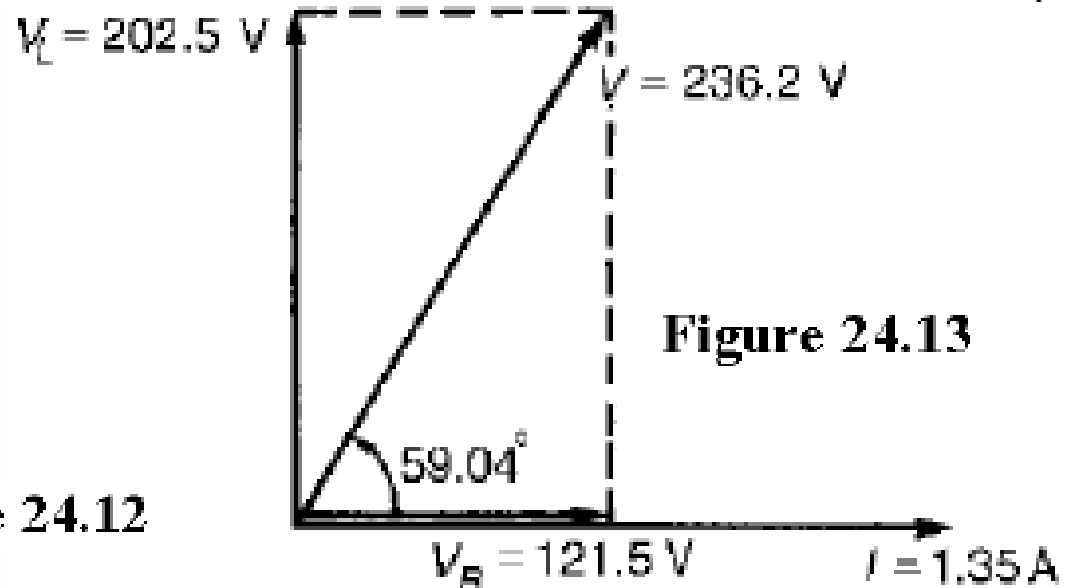
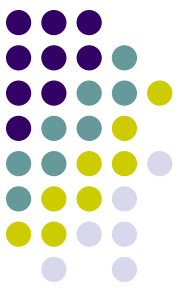


Figure 24.13

The circuit diagram is shown in Figure 24.12



(a) Circuit impedance  $Z = R + jX_L = (90 + j150)\Omega$  or  $174.93 \angle 59.04^\circ \Omega$

Supply voltage,  $V = IZ = (1.35 \angle 0^\circ)(174.93 \angle 59.04^\circ)$   
 $= 236.2 \angle 59.04^\circ \text{ V or } (121.5 + j202.5)\text{V}$

(b) Voltage across  $90 \Omega$  resistor,  $V_R = 121.5 \text{ V}$  (since  $V = V_R + jV_L$ )

(c) Voltage across inductance,  $V_L = 202.5 \text{ V}$  leading  $V_R$  by  $90^\circ$ .

(d) Circuit phase angle is the angle between the supply current and voltage, i.e.,  $59.04^\circ$  lagging (i.e., current lags voltage). The phasor diagram is shown in Figure 24.13.