ITEGRAL GARIS

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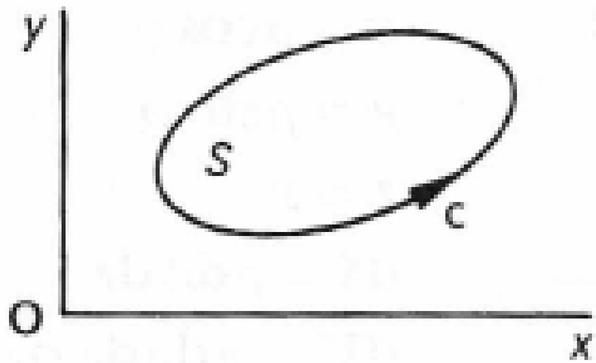
Teorema Green's

Green's theorem



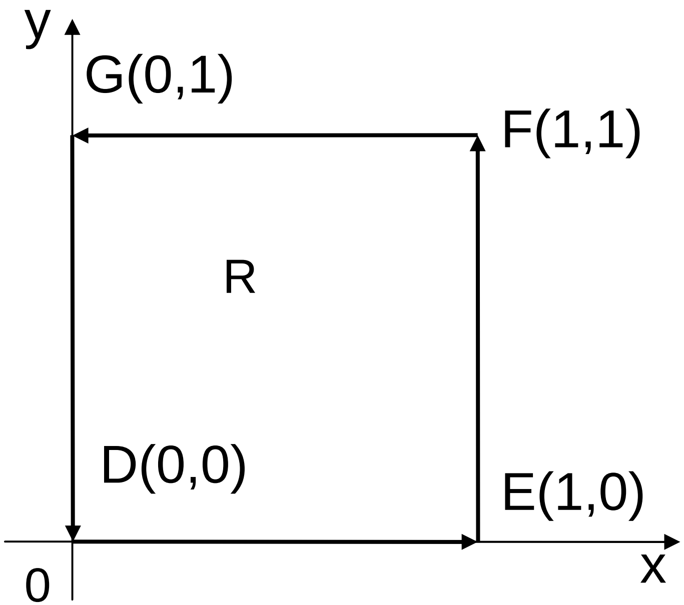
The curve c is a simple closed curve enclosing a plane space S in the x–y plane. P and Q are functions of both x and y.

Then $$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y).$$

Contoh 2 :

a. Evaluate $\oint_R xydx + x^2 dy$ around the sides of the square with vertices D(0,0), E(1,0), F(1,1) and G90,1)

b. Convert the line integral to doble integral and verify Green's thoerem.

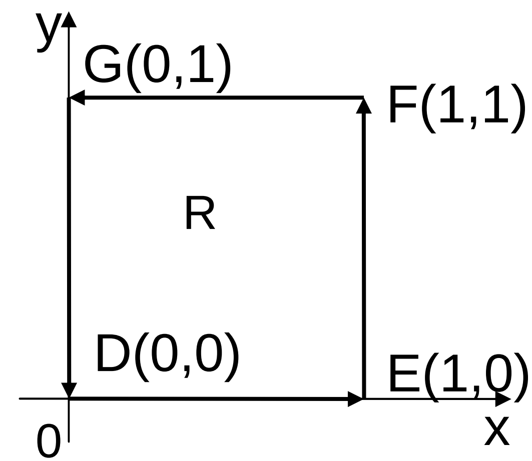


On DE, $y = 0$, $dy = 0$ and $0 \le x \le 1$

On EF, $x = 1$, $dx = 0$ and $0 \le y \le 1$

On FG, $y = 1$, $dy = 0$ and x decreases from 1 to 0

On GD, $x = 0$, $dx = 0$ and y decreases from 1 to 0



$$\oint_C = \int_D^E + \int_E^F + \int_F^G + \int_G^D$$

$$\oint_C xy\,dx + x^2 dy = 0 + \int_0^1 1\,dy + \int_1^0 x\,dx + 0$$

$$= \left[y\right]_0^1 + \left[\frac{x^2}{2}\right]_1^0 = 1 - \frac{1}{2} = \frac{1}{2}$$

Teorema Green's

$$P(x,y) = xy \qquad \text{and} \qquad Q(x,y) = x^2$$

$$\frac{\partial Q}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial P}{\partial y} = x$$

$$\oint_C xydx + x^2dy = \iint_R (2x - x)dxdy$$

$$= \int_0^1 \int_0^1 xdxdy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^1 = \int_0^1 \frac{1}{2}dy$$

$$= \left[\frac{1}{2}y\right]_0^1 = \frac{1}{2}$$

Integral garis

Scalar field

If a scalar field V exists for all points on the curve, then $\sum_{p=1}^{n} V \, d\mathbf{r}_p$ with $d\mathbf{r} \to 0$, defines the *line integral* of V along the curve c from A to B,

$$\text{i.e. line integral} = \int_c V \, d\mathbf{r}$$

Contoh 2 :

If $V = xy^2z$, evaluate $\int_c V \, d\mathbf{r}$ along the curve c having parametric equations $x = 3u$; $y = 2u^2$; $z = u^3$ between A (0, 0, 0) and B (3, 2, 1).

$$V = xy^2z = (3u)(4u^4)(u^3) = 12u^8$$

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = \ldots\ldots\ldots\ldots$$

$$d\mathbf{r} = \mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du$$

Because

$x = 3u, \quad \therefore \; dx = 3\,du$

$y = 2u^2, \quad \therefore \; dy = 4u\,du$

$z = u^3, \quad \therefore \; dz = 3u^2\,du$

Limits: A (0, 0, 0) corresponds to $u =$ . . . .

B (3, 2, 1) corresponds to $u =$ . . . .

$$A\,(0,0,0) \equiv u = 0 \qquad B\,(3,2,1) \equiv u = 1$$

$$\therefore \int_c V\,d\mathbf{r} = \int_0^1 12u^8\,(\mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du)$$

$$4\mathbf{i} + \frac{24}{5}\mathbf{j} + \frac{36}{11}\mathbf{k}$$

Because

$$\int_c V\,d\mathbf{r} = 12\int_0^1 (\mathbf{i}\,3u^8\,du + \mathbf{j}\,4u^9\,du + \mathbf{k}\,3u^{10}\,du)$$

Vector field

To evaluate the line integral, $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ are expressed in terms of x, y, z and the curve in parametric form. We have

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

and $$\mathrm{d}\mathbf{r} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z$$

Then $$\mathbf{F}\cdot\mathrm{d}\mathbf{r} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z)$$
$$= F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

$$\therefore \int_c \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_c F_x\,\mathrm{d}x + \int_c F_y\,\mathrm{d}y + \int_c F_z\,\mathrm{d}z$$

Contoh 2 :

If $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} - 2yz\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ between A (0, 0, 0) and B (4, 2, 1) along the curve having parametric equations $x = 4t$; $y = 2t^2$; $z = t^3$.

Expressing everything in terms of the parameter t, we have

$$\mathbf{F} = \ldots\ldots\ldots\ldots$$

$$\mathrm{d}x = \ldots\ldots\ldots\ldots; \quad \mathrm{d}y = \ldots\ldots\ldots\ldots; \quad \mathrm{d}z = \ldots\ldots\ldots\ldots$$

$$\mathbf{F} = 32t^4\,\mathbf{i} + 4t^4\,\mathbf{j} - 4t^5\,\mathbf{k}$$
$$dx = 4\,dt; \quad dy = 4t\,dt; \quad dz = 3t^2\,dt$$

Because

$$x^2y = (16t^2)(2t^2) = 32t^4 \qquad x = 4t \qquad \therefore\ dx = 4\,dt$$
$$xz = (4t)(t^3) = 4t^4 \qquad y = 2t^2 \qquad \therefore\ dy = 4t\,dt$$
$$2yz = (4t^2)(t^3) = 4t^5 \qquad z = t^3 \qquad \therefore\ dz = 3t^2\,dt$$

Then $\int \mathbf{F}\cdot d\mathbf{r} = \int (32t^4\mathbf{i} + 4t^4\mathbf{j} - 4t^5\mathbf{k})\cdot(\mathbf{i}\,4\,dt + \mathbf{j}\,4t\,dt + \mathbf{k}\,3t^2\,dt)$

$$= \int (128t^4 + 16t^5 - 12t^7)\,dt$$

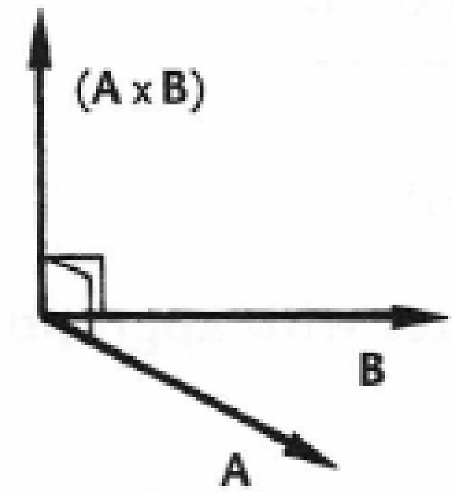
Limits: A (0, 0, 0) ≡ t = ...; B (4, 2, 1) ≡ t = ...

$$A \equiv t = 0; \quad B \equiv t = 1$$

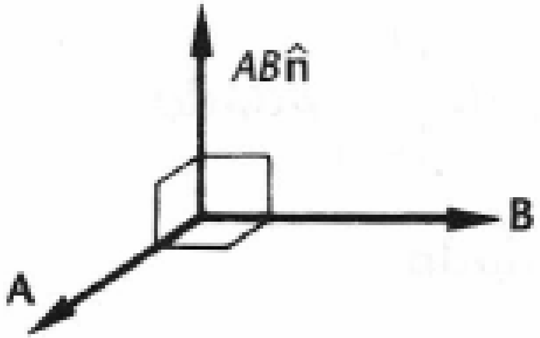
$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (128t^4 + 16t^5 - 12t^7)\,dt = \ldots\ldots\ldots\ldots$$

$$\frac{128}{5} + \frac{8}{3} - \frac{3}{2} = \frac{803}{30} = 26{\cdot}77$$

Integral Permukaan

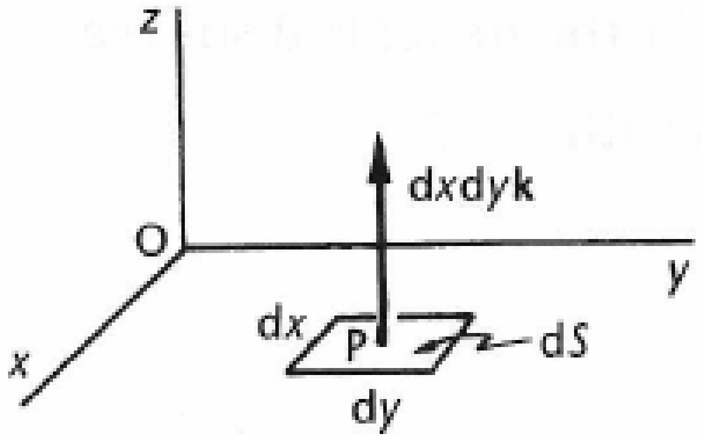


The vector product of two vectors **A** and **B** has magnitude $|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$ at right angles to the plane of **A** and **B** to form a right-handed set.



If $\theta = \frac{\pi}{2}$, then $|\mathbf{A} \times \mathbf{B}| = AB$ in the direction of the normal. Therefore, if $\hat{\mathbf{n}}$ is a unit normal then

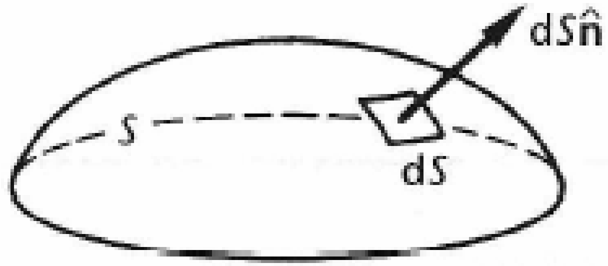
$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\hat{\mathbf{n}} = AB\,\hat{\mathbf{n}}$$



If P (x, y) is a point in the x–y plane, the element of area $dx\,dy$ has a vector area $d\mathbf{S} = (\mathbf{i}\,dx) \times (\mathbf{j}\,dy)$.

i.e. $d\mathbf{S} = dx\,dy(\mathbf{i} \times \mathbf{j}) = dx\,dy\,\mathbf{k}$

i.e. a vector of magnitude $dx\,dy$ acting in the direction of $\mathbf{k}$ and referred to as the *vector area*.



For a general surface S in space, each element of surface dS has a *vector area* dS such that $d\mathbf{S} = dS\,\hat{\mathbf{n}}$.

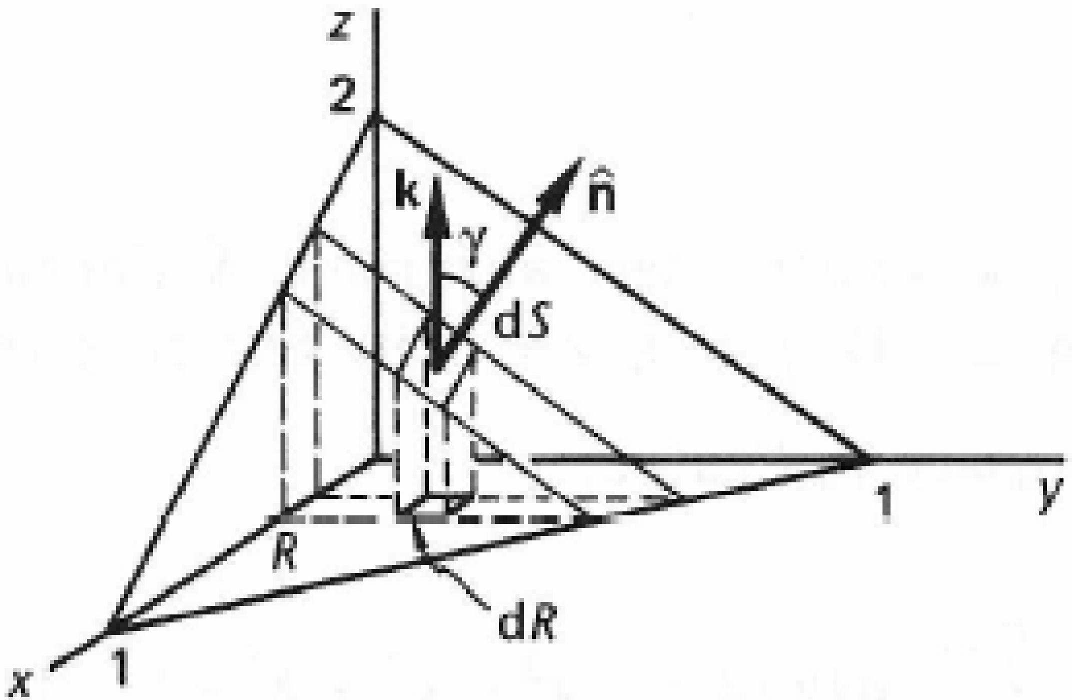
You will remember we established previously that for a surface S given by the equation $\phi(x, y, z) = \text{constant}$, the unit normal $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$$

Contoh 2 :

A scalar field $V = x + y + z$ exists over the surface S defined by $2x + 2y + z = 2$ bounded by $x = 0,\ y = 0,\ z = 0$ in the first octant.

Evaluate $\int_S V \, d\mathbf{S}$ over this surface.



$$S:\quad 2x + 2y + z = 2$$

$$x = 0 \qquad z = 2 - 2y$$

$$y = 0 \qquad z = 2 - 2x$$

$$z = 0 \qquad y = 1 - x$$

$$d\mathbf{S} = \hat{\mathbf{n}}\, dS \qquad \text{where} \quad \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

Therefore

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3} \quad \text{so that} \quad d\mathbf{S} = \hat{\mathbf{n}}\, dS = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dS$$

If we now project dS onto the x–y plane, $dR = dS\cos\gamma$

$$\cos\gamma = \hat{\mathbf{n}} \cdot \mathbf{k} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{k}) = \frac{1}{3}$$

$$\therefore\ dR = \frac{1}{3}dS \qquad \therefore\ dS = 3dR = 3\,dx\,dy$$

$$\therefore\ \int_S V\, d\mathbf{S} = \int_S V\hat{\mathbf{n}}\, dS = \int_S\int (x+y+z)\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})3\,dx\,dy$$

$$\int_S V\,d\mathbf{S} = \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^{1-x} (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\,dx$$

$$= \left[\frac{3}{2}x - x^2 + \frac{x^3}{6}\right]_0^1 (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$= \frac{2}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Contoh 2 :

Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}$ is the field $x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}$ and S is the surface $2x + y + 2z = 2$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant.

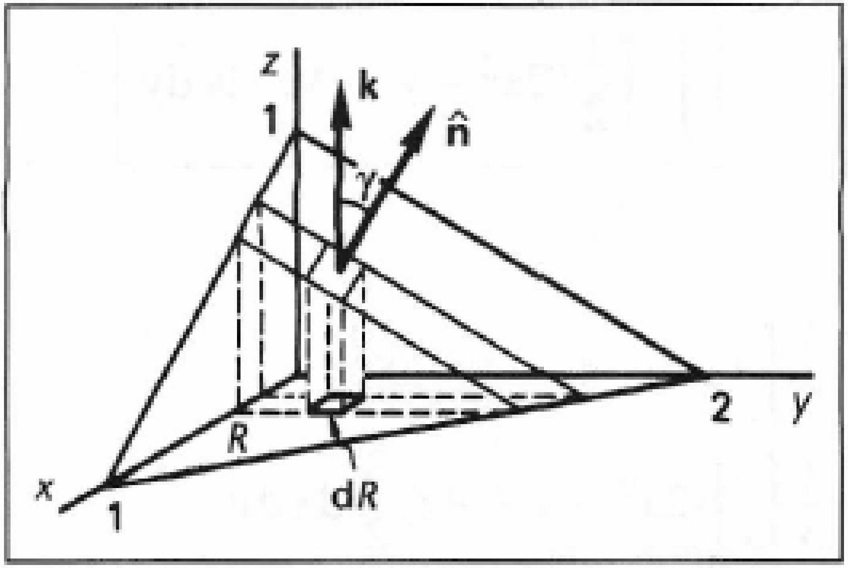
We can sketch the diagram by putting $x = 0$, $y = 0$, $z = 0$ in turn in the equation for S.

When $x = 0 \quad y + 2z = 2 \quad z = 1 - \frac{y}{2}$

$y = 0 \quad x + z = 1 \quad z = 1 - x$

$z = 0 \quad 2x + y = 2 \quad y = 2 - 2x$

So the diagram is



$$\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}; \qquad \phi: \quad 2x + y + 2z - 2 = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \qquad |\nabla\phi| = 3$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S (x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k})\cdot\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})\,dS$$
$$= \frac{1}{3}\int_S (2x^2 - y + 4z)\,dS$$

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{3}\int_S (2x^2 - y + 4z)\,dS$$
$$= \frac{1}{3}\int_R\int (2x^2 - y + 4z)\frac{3}{2}\,dx\,dy$$
$$= \frac{1}{2}\int_R\int (2x^2 - y + 4z)\,dx\,dy$$

Limits: $y = 0$ to $y = 2 - 2x$; $x = 0$ to $x = 1$

$$\therefore \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_0^1\int_0^{2-2x}(2x^2 - y + 4z)\,dy\,dx$$

But $2x + y + 2z = 2 \quad \therefore \quad z = \frac{1}{2}(2 - 2x - y)$

$$\therefore \quad \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \ldots\ldots\ldots\ldots$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{1}{2}\int_0^1 \int_0^{2-2x} (2x^2 - y + 4 - 4x - 2y)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{2-2x} (2x^2 - 4x + 4 - 3y)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[(2x^2 - 4x + 4)y - \frac{3y^2}{2}\right]_0^{2-2x} dx$$

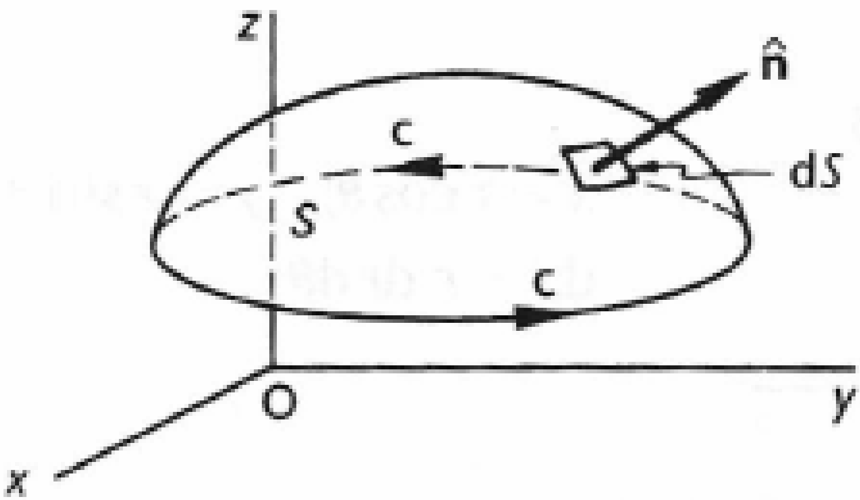
$$= \frac{1}{2}\int_0^1 (4x^2 - 8x + 8 - 4x^3 + 8x^2 - 8x - 6 + 12x - 6x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 (6x^2 - 4x^3 - 4x + 2)\, dx = \int_0^1 (3x^2 - 2x^3 - 2x + 1)\, dx$$

$$= \left[x^3 - \frac{x^4}{2} - x^2 + x\right]_0^1 = \frac{1}{2}$$

Teorema Stoke's

Stokes' theorem



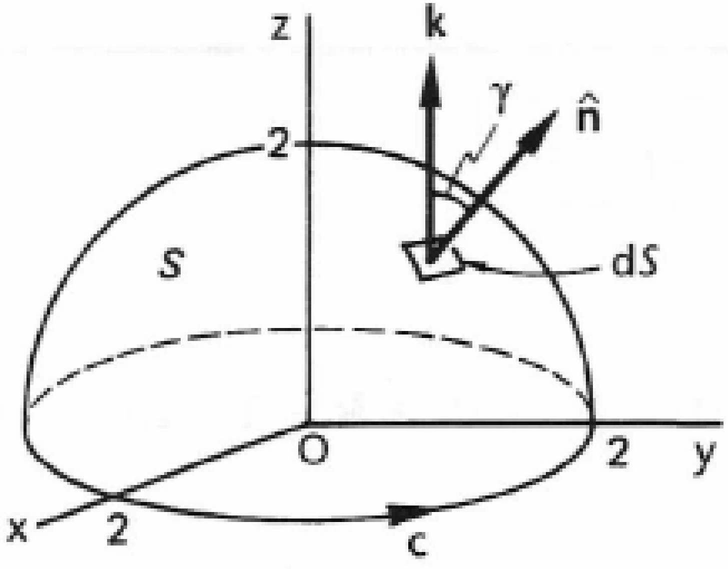
An open surface S bounded by a simple closed curve c, then

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Contoh 2 :

A hemisphere S is defined by $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$. A vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$ exists over the surface and around its boundary c.

Verify Stokes' theorem, that $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$.



S: $x^2 + y^2 + z^2 - 4 = 0$

$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$

c is the circle $x^2 + y^2 = 4$.

(a) $$\oint_c \mathbf{F}\cdot d\mathbf{r} = \int_c (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= \int_c (2y\,dx - x\,dy + xz\,dz)$$

Converting to polar coordinates

$x = 2\cos\theta;$ $y = 2\sin\theta;$ $z = 0$

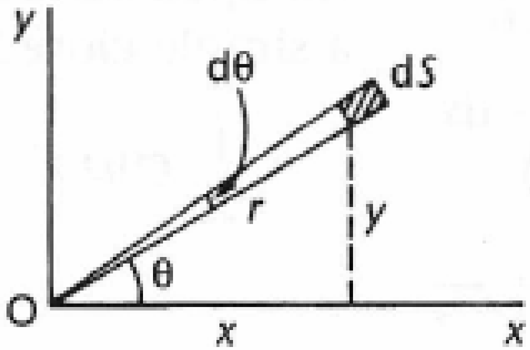
$dx = -2\sin\theta\,d\theta;$ $dy = 2\cos\theta\,d\theta;$ Limits $\theta = 0$ to 2π

Making the substitutions and completing the integral

$$\oint_c \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

Polar coordinates

(a) Plane polar coordinates (r, θ)



$x = r\cos\theta;$ $y = r\sin\theta$

$dS = r\,dr\,d\theta$

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (4\sin\theta[-2\sin\theta\, d\theta] - 2\cos\theta\, 2\cos\theta\, d\theta)$$

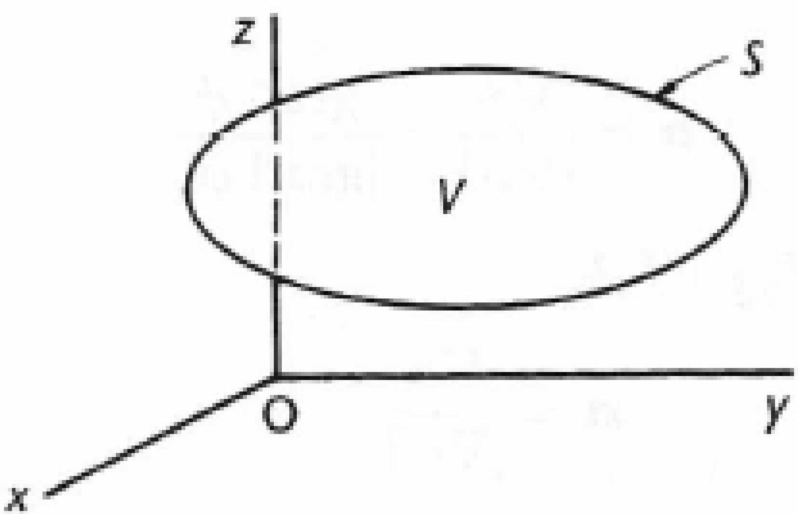
$$= -4\int_0^{2\pi} (2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= -4\int_0^{2\pi} (1 + \sin^2\theta)\, d\theta = -2\int_0^{2\pi} (3 - \cos 2\theta)\, d\theta$$

$$= -2\left[3\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -12\pi$$

Teorema Gauss

Divergence theorem (Gauss' theorem)



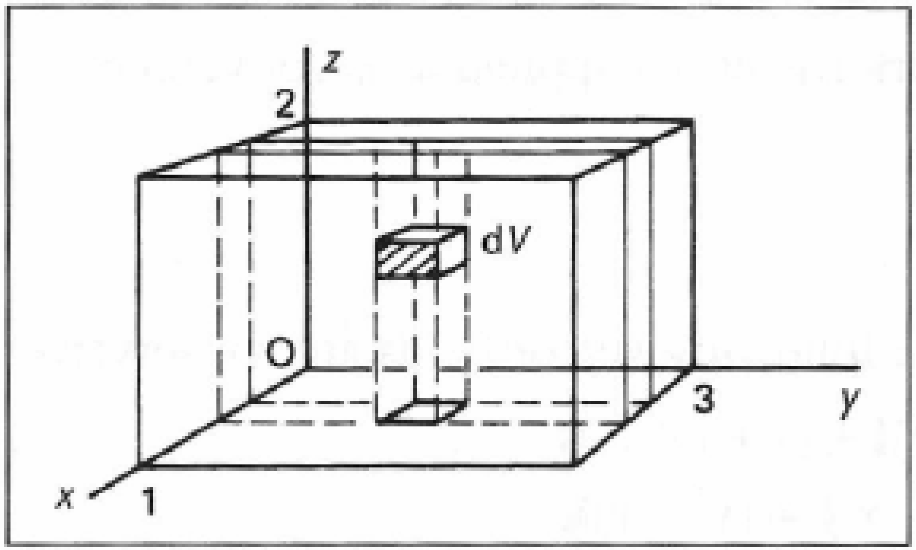
Closed surface S enclosing a region V in a vector field $\mathbf{F}$.

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

Contoh 2 :

Verify the divergence theorem for the vector field $\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ taken over the region bounded by the planes $z = 0$, $z = 2$, $x = 0$, $x = 1$, $y = 0$, $y = 3$.

Start off, as always, by sketching the relevant diagram, which is



$$dV = dx\,dy\,dz$$

We have to show that

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

(a) To find $\int_V \operatorname{div} \mathbf{F}\, dV$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\, \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x + 0 + 0 = 2x$$

$$\therefore \quad \int_V \operatorname{div} \mathbf{F}\, dV = \int_V 2x\, dV = \iiint_V 2x\, dz\, dy\, dx$$

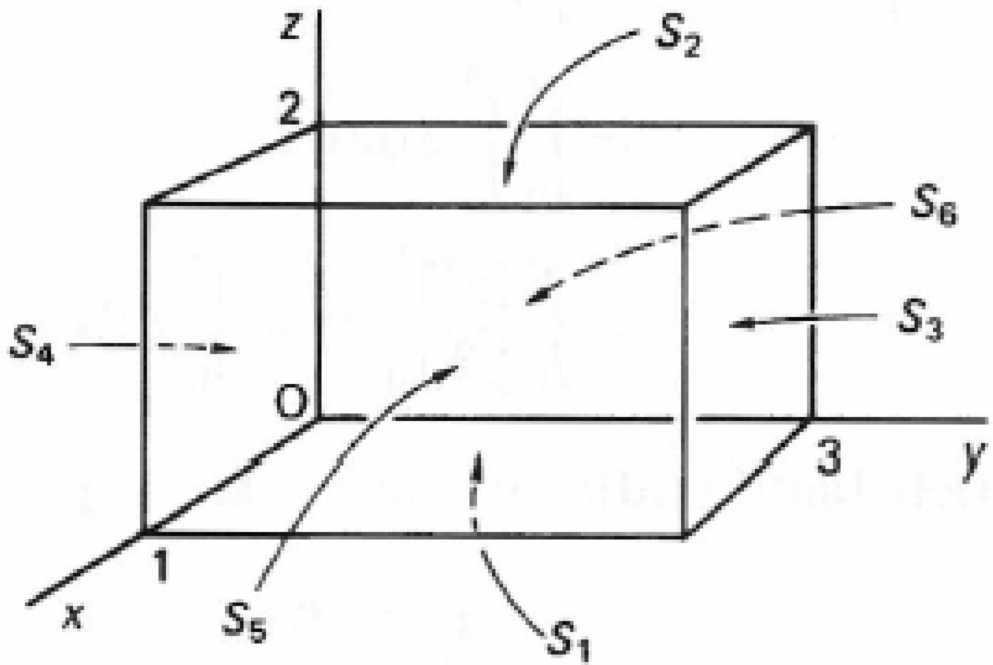
Inserting the limits and completing the integration

$$\int_V \operatorname{div} \mathbf{F}\, dV = \ldots\ldots\ldots\ldots$$

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^3 \int_0^2 2x\, dz\, dy\, dx = \int_0^1 \int_0^3 \Big[2xz\Big]_0^2 dy\, dx$$

$$= \int_0^1 \Big[4xy\Big]_0^3 dx = \int_0^1 12x\, dx = \Big[6x^2\Big]_0^1 = 6$$

(b) To find $\int_S \mathbf{F} \cdot d\mathbf{S}$ i.e. $\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$



The enclosing surface S consists of six separate plane faces denoted as $S_1, S_2, \ldots, S_6$ as shown. We consider each face in turn.

$$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$$

(1) S_1 (base): $z = 0$; $\hat{\mathbf{n}} = -\mathbf{k}$ (outwards and downwards)

$$\therefore \quad \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \qquad dS_1 = dx\,dy$$

$$\therefore \quad \int_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} (x^2\mathbf{i} + y\mathbf{k})\cdot(-\mathbf{k})\,dy\,dx$$

$$= \int_0^1 \int_0^3 (-y)\,dy\,dx$$

$$= \int_0^1 \left[-\frac{y^2}{2}\right]_0^3 dx$$

$$= -\frac{9}{2}$$

(2) S_2 (top): $z = 2$; $\hat{\mathbf{n}} = \mathbf{k}$ $dS_2 = dx\,dy$

$$\therefore \quad \int_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

$$\int_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} (x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k})\cdot(\mathbf{k})\,dy\,dx$$

$$= \int_0^1 \int_0^3 y\,dy\,dx = \frac{9}{2}$$

(3) S_3 (right-hand end): $y = 3;$ $\quad \hat{\mathbf{n}} = \mathbf{j}$ $\quad dS_3 = dx\,dz$

$$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$$

$$\therefore \int_{S_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int\int_{S_3} (x^2\mathbf{i} + z\mathbf{j} + 3\mathbf{k})\cdot(\mathbf{j})\,dz\,dx$$

$$= \int_0^1 \int_0^2 z\,dz\,dx$$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx = \int_0^1 2\,dx = 2$$

(4) S_4 (left-hand end): $y = 0;$ $\quad \hat{\mathbf{n}} = -\mathbf{j}$ $\quad dS_4 = dx\,dz$

$$\therefore \int_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

$$\int_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int\int_{S_4} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})\cdot(-\mathbf{j})\,dz\,dx = \int_0^1 \int_0^2 (-z)\,dz\,dx$$

$$= \int_0^1 \left[-\frac{z^2}{2}\right]_0^2 dx = \int_0^1 (-2)\,dx = -2$$

(5) S_5 (front): $x = 1;$ $\hat{\mathbf{n}} = \mathbf{i}$ $dS_5 = dy\,dz$

$$\therefore \int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_5} (\mathbf{i} + z\mathbf{j} + y\mathbf{k})\cdot(\mathbf{i})\,dy\,dz = \iint_{S_5} 1\,dy\,dz = 6$$

(6) S_6 (back): $x = 0;$ $\hat{\mathbf{n}} = -\mathbf{i}$ $dS_6 = dy\,dz$

$$\therefore \int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_6} (z\mathbf{j} + y\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = \iint_{S_6} 0\,dy\,dz = 0$$

For the whole surface S we therefore have

$$\int_S \mathbf{F}\cdot d\mathbf{S} = -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6$$

and from our previous work in section (a) $\int_V \operatorname{div}\mathbf{F}\,dV = 6$

We have therefore verified as required that, in this example

$$\int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$